CARINGBAH HIGH SCHOOL

2020 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using blue or black pen
- Calculators approved by NESA may be used
- A reference sheet is provided on separate paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–10)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Marker's Use Only						
Section I	**Section II**				**Total**	
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**		
/10	/15	/15	/15	/15	/70	%

Section I

10 marks
Attempt Questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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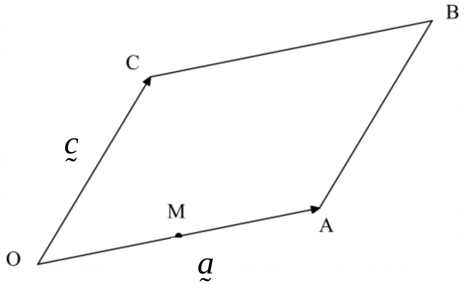
1 Which of the following pairs of parametric equations represents a circle that passes through the origin?

(A) $x = 3 + 3\cos\theta,\ y = 4 + 3\sin\theta$
(B) $x = 3 + 4\cos\theta, y = 4 + 4\sin\theta$
(C) $x = 3 + 5\cos\theta,\ y = 4 + 5\sin\theta$
(D) $x = 3 + 7\cos\theta,\ y = 4 + 7\sin\theta$

2 A spherical balloon is being inflated at a constant rate of 200π cm^3 s^{-1}. At what rate is the radius of the balloon increasing when the radius is 10 cm?

(A) 0.25 cm s^{-1}
(B) 0.5 cm s^{-1}
(C) 1.0 cm s^{-1}
(D) 2.0 cm s^{-1}

3 In the diagram, *OABC* is parallelogram. The vector $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. *M* is the midpoint of *OA*. Which of the following expressions is represented by the vector $\overrightarrow{MB}$?



(A) $\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{c}$

(B) $\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{c}$

(C) $\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}$

(D) $\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$

4 What is the number of distinct solutions of the equation $3\cos\theta + 4\sin\theta = 5$ for $0 \le \theta \le 2\pi$?

(A) 1
(B) 2
(C) 3
(D) 4

5 Solve the inequality $\dfrac{x^2 - 4}{x} \ge 0$.

(A) $-2 \le x < 0$ or $x \ge 2$
(B) $-2 \ge x > 0$ or $x \le 2$
(C) $-4 \le x < 0$ or $x \ge 4$
(D) $-4 \ge x > 0$ or $x \le 4$

6 Which of the following vectors is perpendicular to the vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$?

(A) $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$

(B) $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

(C) $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$

(D) $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$

7 What is the value of $\dfrac{d}{dx}\sin^{-1}\sqrt{1-x^2}$?

(A) $\dfrac{-1}{x\sqrt{1-x^2}}$

(B) $\dfrac{-1}{\sqrt{1-x^2}}$

(C) $\dfrac{1}{\sqrt{1-x^2}}$

(D) $\dfrac{1}{x\sqrt{1-x^2}}$

8 What is the value of k such that the function $f(x)=\dfrac{k}{1+x^2}$, $-1\le x\le 1$ is a probability density function?

(A) $k=\dfrac{\pi}{4}$

(B) $k=\dfrac{\pi}{2}$

(C) $k=\dfrac{2}{\pi}$

(D) $k=\dfrac{4}{\pi}$

9 A geometric series $1-\dfrac{1}{x}+\dfrac{1}{x^2}-\dfrac{1}{x^3}+\ldots$ has limiting sum S.

For what values of x is $S<1$?

(A) $x>-1$

(B) $x>-1$, $x\neq 0$

(C) $x>0$

(D) $x>1$

10 Three fair dice of different colours are rolled together. What is the probability that the product of the three scores is a perfect square?

(A) $\dfrac{6}{216}$

(B) $\dfrac{13}{216}$

(C) $\dfrac{32}{216}$

(D) $\dfrac{38}{216}$

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
In Question 11-14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use the Question 11 Writing Booklet.**

(a) Differentiate $\tan^{-1}(\log_e x)$ **1**

(b) Find $\int_0^{\frac{\pi}{6}} \sin x \cos x \, dx$ **2**

(c) (i) Find the unit vector in the direction of the vector $(4\underset{\sim}{i} + 3\underset{\sim}{j})$. **1**

(ii) A particle is moving with velocity $v = (15\underset{\sim}{i} + 20\underset{\sim}{j})\, ms^{-1}$.
Find the component of v in the direction of the vector $(4\underset{\sim}{i} + 3\underset{\sim}{j})$. **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to find the solutions of the equation **3**
$\cos x - 3\sin x + 3 = 0$ for $0 \le x \le 2\pi$ correct to 2 decimal places

(e) The discrete random variable X has a binomial distribution with n independent trials with probability of success p. If its mean is 6 and its standard deviation is 2, find the values of n and p. **3**

(f) Use the mathematical induction to prove that $3^{2n+4} - 2^{2n}$ is divisible by 5 for integers $n \ge 1$. **3**

End of Question 11

Question 12 (15 marks) **Use the Question 12 Writing Booklet.**

(a) (i) Sketch the graph of the function $y = \tan^{-1}\frac{1}{2}(x-2)$ showing clearly the intercepts on the axes and the equations of the asymptotes. **2**

(ii) The graph of the function $y = \tan^{-1}\frac{1}{2}(x-2)$ is transformed by a translation left by 1 unit, then a horizontal dilation with a scale factor 2. **2**

Find the equation of the transformed graph.

(b) (i) Records show that 64% of students at a school travelled to and from school by bus. Samples of 100 students at the school are taken to determine the proportion who travel to and from school by bus. Show that the distribution of such sample proportions has mean 0.64 and standard deviation 0.048. **2**

(ii) Use the table (shown below) of $P(Z < z)$, where Z has a standard normal distribution, to estimate the probability that a sample of 100 students will contain at least 58 and at most 64 students who travel to and from school by bus. **3**

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

(c) The polynomial equation $4x^3 - 12x^2 + 5x + 6 = 0$ has roots α, β and γ. It is known that one of the roots is the sum of the other two. Find α, β and γ. **3**

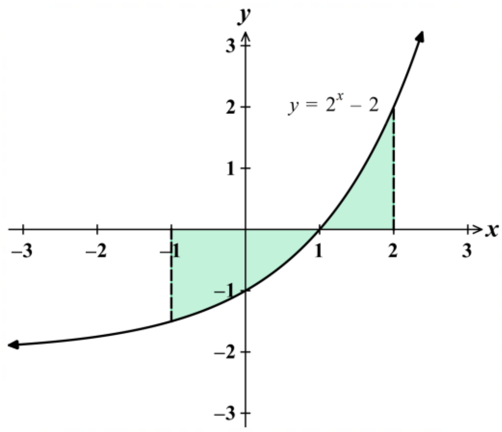
(d) By using the fact that $(1+x)^{11} = (1+x)^3(1+x)^8$. Show that **3**

$$\binom{11}{5} = \binom{8}{5} + \binom{3}{1}\binom{8}{4} + \binom{3}{2}\binom{8}{3} + \binom{8}{2}$$

End of Question 12

Question 13 (15 marks) **Use the Question 13 Writing Booklet.**

(a) **4**



In the diagram above, the region bounded by the curve $y = 2^x - 2$ and the x axis between $x = -1$ and $x = 2$ is rotated through one revolution about the x axis.

Find in simplest **exact** form the volume of the solid formed.

(b) A vertical tower of height h metres stands with its base at a point O on horizonal ground. At the same instant, stone A is projected from O with speed $V\ ms^{-1}$ at an angle α above the horizontal and stone B is projected from the top of the tower with speed $U\ ms^{-1}$ at an angle β above the horizontal, where $\alpha > \beta$. The two stones move in the same vertical plane under gravity, where the acceleration due to gravity is $g\ ms^{-2}$, and collide after T seconds. At time t seconds after firing, the position vectors of the two stones are **4**

$$r_A(t) = (Vt\cos\alpha)\underset{\sim}{i} + (-\frac{1}{2}gt^2 + Vt\sin\alpha)\underset{\sim}{j}$$

$$r_B(t) = (Ut\cos\beta)\underset{\sim}{i} + (h - \frac{1}{2}gt^2 + Ut\sin\beta)\underset{\sim}{j}$$. (DO NOT PROVE THIS)

Show that $T = \dfrac{h\cos\alpha}{U\sin(\alpha-\beta)}$.

Question 13 continues on page 8

Question 13 (continued)

(c) (i) Prove the trigonometric identity $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ using the compound angle formula. **2**

(ii) Hence find expressions for the exact values of the solutions to the equation $\dfrac{3x - x^3}{1 - 3x^2} = \sqrt{3}$. **3**

(iii) Hence show that $\tan\dfrac{\pi}{9} \times \tan\dfrac{4\pi}{9} \times \tan\dfrac{7\pi}{9} = -\sqrt{3}$. **2**

End of Question 13

Question 14 (15 marks) **Use the Question 14 Writing Booklet.**

(a) The cubic equation $2x^3 - x^2 + x - 1 = 0$ has roots α, β and γ.
Show without finding each individual root that

(i) $\alpha^2 + \beta^2 + \gamma^2 = -\frac{3}{4}$ **2**

(ii) $\alpha^3 + \beta^3 + \gamma^3 = \frac{7}{8}$ **2**

(b) (i) Using the substitution $x = \cos 2\theta$, show **2**

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx = \int_0^{\frac{\pi}{4}} 4\sin^2\theta\, d\theta$$

(ii) Hence evaluate in simplest exact form $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$ **2**

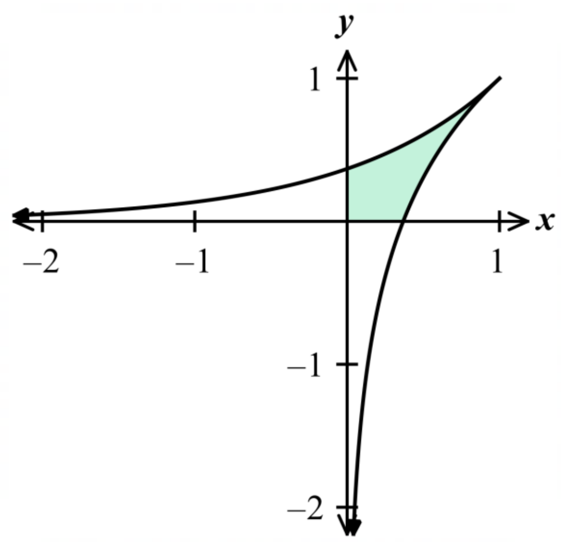
Question 14 continues on page 10

Question 14 (continued)

(c) The function $f(x) = 1 + \ln x$ is defined in the domain $(0,1]$.

(i) Show that $\frac{d}{dx}\left(x \ln x\right) = 1 + \ln x$ **1**

The diagram shows the graphs of the function $y = f(x)$ and the inverse function $y = f^{-1}(x)$.



(ii) Find in simplest exact form the area of the shaded region in the first quadrant bounded by the curves $y = f(x)$, $y = f^{-1}(x)$ and the coordinate axes. **3**

(iii) Sketch the graph of the curve $y = \frac{1}{f(x)}$, showing clearly the coordinates of the endpoints and the equations of any asymptotes. **2**

(iv) Use **interval notation** to state the **range** of the function $y = \frac{1}{f(x)}$ **1**

End of Examination

Student Name/Number ______________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ● B ~~●~~ C ○ D ○

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

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Examination: Answers

Student Number

MCQ

1) C 2) B 3) B 4) A 5) A 6) D

7) B 8) C 9) D 10) D.

Question 11

a) $\frac{d}{dx}\left(\tan^{-1}(\log_e x)\right)$

$= \frac{1}{1+(\log_e x)^2} \times \frac{1}{x}$

$= \frac{1}{x(1+(\log_e x)^2)}$

1 mark for correct answer

b) $\int_0^{\frac{\pi}{6}} \sin x \cos x \, dx$

$= \left[\frac{1}{2}(\sin x)^2\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\left(\sin\frac{\pi}{6}\right)^2 - 0$

$= \frac{1}{2} \times \frac{1}{4}$

$= \frac{1}{8}$

1 mark for working

1 mark for correct answer

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Student Number

Examination:.................................

(c) (i) Unit Vector :

$\sqrt{4^2+3^2} = 5$

1 mark for correct ans

Hence required unit vector is $\frac{4}{5}\underset{\sim}{i} + \frac{3}{5}\underset{\sim}{j}$

(ii) $\begin{pmatrix} 15 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$

1 mark for correct dot product

1 mark for correct final answer

$= 12 + 12$

$= 24$

Required component of $\underset{\sim}{v}$ is $24\left(\frac{4}{5}\underset{\sim}{i} + \frac{3}{5}\underset{\sim}{j}\right)$

$= \frac{96}{5}\underset{\sim}{i} + \frac{72}{5}\underset{\sim}{j}$

(d) let $t = \tan\frac{x}{2}$

$\sin x = \frac{2t}{1+t^2}$ $\quad \cos x = \frac{1-t^2}{1+t^2}$ $\quad 0 \le x \le 2\pi$

$\cos x - 3\sin x + 3 = 0$

$\frac{1-t^2}{1+t^2} - \frac{6t}{1+t^2} + 3 = 0$

$1 - t^2 - 6t + 3(1+t^2) = 0$

$1 - t^2 - 6t + 3 + 3t^2 = 0$

$2t^2 - 6t + 4 = 0$

$t^2 - 3t + 2 = 0$

$(t-1)(t-2) = 0$

$t = 1 \quad t = 2$

For $\tan\frac{x}{2} = 1$ $\quad \tan\frac{x}{2} = 2$

$\frac{x}{2} = \tan^{-1}(1)$ $\quad \frac{x}{2} = \tan^{-1}(2)$

$\frac{x}{2} = \frac{\pi}{4}$ $\quad \frac{x}{2} = 1.107$

$x = \frac{\pi}{2}$ $\quad x = 2.21$

$\therefore x = 2.21,\ \frac{\pi}{2}(1.57)$

1 mark for obtaining correct simplified quadratic equation in t

1 mark for $x = \frac{\pi}{2}(1.57)$, 1 mark for $x = 2.2$

2

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Examination:……………………………

Student Number

(e) $\mu = 6 \Rightarrow np = 6$ ①

$\sigma^2 = 4 \Rightarrow npq = 4$ ②

1 mark for writing expressions for μ and σ in terms of n, p, q.
1 mark for $p = \frac{1}{3}$
1 mark for $n = 18$

② ÷ ①

$q = \frac{2}{3}$

$\therefore p = \frac{1}{3}$ and $n = 18$

1 mark for showing truth $n = 1$
2 marks for understanding and applying the induction process

(f) Step 1: Let $n = 1$

$3^{2+4} - 2^{2} = 725 = 5 \times 145$

$\therefore$ the statement is true for $n = 1$

Step 2: Assume this statement is true for $n = k$

i.e. $3^{2k+4} - 2^{2k} = 5M$ for some integer M.

RTP the statement is true for $n = k+1$

RTP $3^{2(k+1)+4} - 2^{2(k+1)}$ is divisible by 5

$= 3^2 \times 3^{2k+4} - 2^2 \times 2^{2k}$

$= 9 \times (5M + 2^{2k}) - 4 \times 2^{2k}$

$= 45M + 9 \times 2^{2k} - 4 \times 2^{2k}$

$= 45M + 5 \times 2^{2k}$

$= 5 \times (9M + 2^{2k})$ where $9M + 2^{2k}$ is integral

3

Step 3: Hence if $n = k$ is true, then the statement is true for $n = k+1$

$\therefore$ the statement is true for all integers $n \geq 1$

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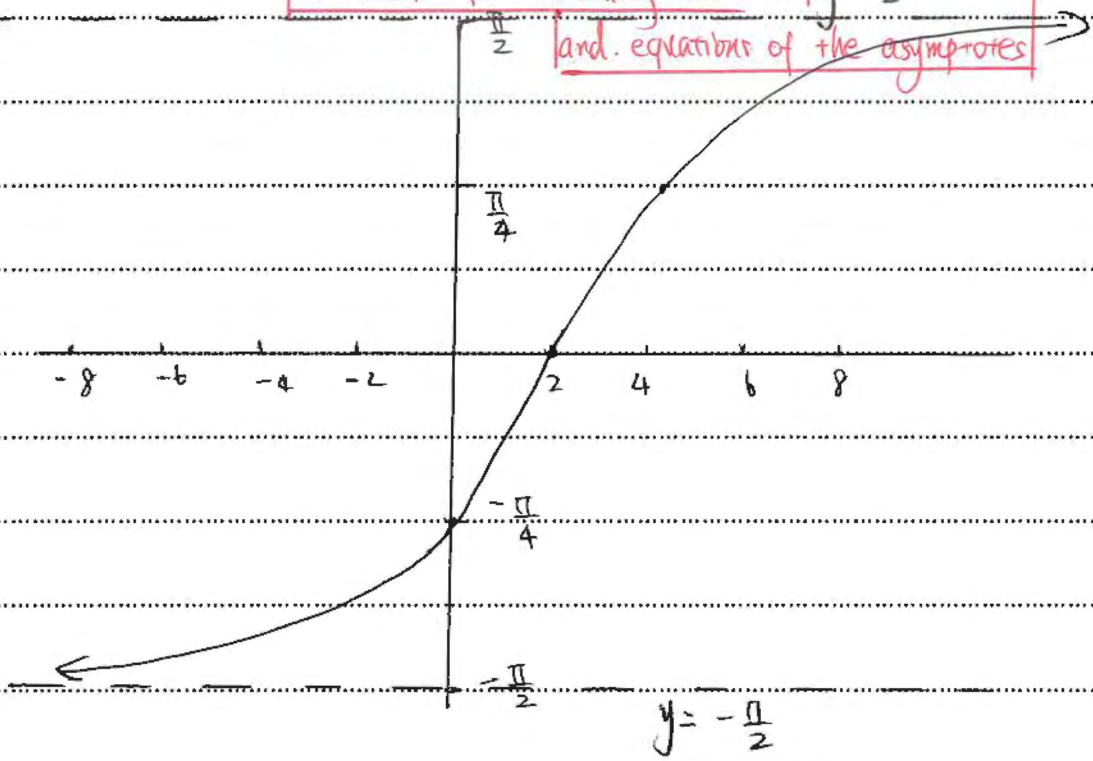
Student Number

Examination:.................................

Question 12

1 mark for sketching graph correctly
1 mark for labelling the intercept $y = \frac{\pi}{2}$ and equations of the asymptotes

a) (i)



(ii) Translation left by 1 unit

$x \to x+1 \Rightarrow y = \tan^{-1}\left(\frac{1}{2}(x+1)-2\right)$

$y = \tan^{-1}\frac{1}{2}(x-1)$

Then horizontal dilation by scale factor 2:

1 mark for translation left
1 mark for horizontal dilation

$x \to \frac{1}{2}x \Rightarrow y = \tan^{-1}\frac{1}{2}\left(\frac{1}{2}x-1\right)$

$y = \tan^{-1}\frac{1}{4}(x-2)$

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Examination:..................................

Student Number

(b) For students in the sample, the number of students travelling to and from school by bus is a random variable X with the Binomial distribution $B(100, 0.64)$. Hence the sample proportion $\frac{X}{100}$ has mean $\mu = 0.64$ (since $E\left(\frac{X}{n}\right) = \frac{np}{n} = p$)

and standard deviation $\sigma = \sqrt{\frac{0.64 \times (1-0.64)}{100}}$

$= 0.048$

(since $VAR\left(\frac{X}{n}\right) = \frac{npq}{n^2} = \frac{pq}{n}$)

1 mark for correct explanation to find μ

1 mark for applying the appropriate formula to find σ

(iii)

$P\left(0.58 \le \frac{X}{100} \le 0.64\right)$

Z scores for $0.64 = \frac{0.64 - 0.64}{0.048} = 0$

Z scores for $0.58 = \frac{0.58 - 0.64}{0.048} = -1.25$

$\therefore \approx P(-1.25 \le Z \le 0)$

$= P(Z \le 1.25) - 0.5$

$= 0.8944 - 0.5$

$= 0.3944$

1 mark for calculating the z scores of the interval limits

2 marks for using the normal approx to the distribution of the sample proportion to estimate the required probability from the table

5

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Student Number

Examination:.................................

(c) $4x^3 - 12x^2 + 5x + 6 = 0$

Let α β and γ be the roots of this equation

$\alpha + \beta + \gamma = 3$ ① $\quad$ $\alpha = \beta + \gamma$ ④

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{5}{4}$ ②

$\alpha\beta\gamma = -\frac{3}{2}$ ③

1 mark for correct $\alpha+\beta+\gamma = 3$
$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{5}{4}$ and $\alpha\beta\gamma = -\frac{3}{2}$.
2 marks for correct working out
as $\alpha = \beta+\gamma$ and correct answers

sub ④ into ①

$2\alpha = 3$

$\alpha = \frac{3}{2}$

sub $\alpha = \frac{3}{2}$ into ③

$\frac{3}{2}\beta\gamma = -\frac{2}{3}$

$\beta\gamma = -1$ ⑤ $\quad$ and $\quad$ $\beta + \gamma = \frac{3}{2}$ ⑥

From ⑥ $\beta = \frac{3}{2} - \gamma$

sub β into ⑤

$\left(\frac{3}{2} - \gamma\right)\gamma = -1$

$\frac{3}{2}\gamma - \gamma^2 = -1$ $\qquad$ $\therefore \alpha = \frac{3}{2}$

$\gamma^2 - \frac{3}{2}\gamma - 1 = 0$ $\qquad$ $\beta = 2$

$2\gamma^2 - 3\gamma - 2 = 0$ $\qquad$ $\gamma = -\frac{1}{2}$

$(\gamma - 2)(2\gamma + 1) = 0$

$\gamma = 2 \quad \gamma = \frac{-1}{2}$

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Examination:.....................................

Student Number

(d) $(1+x)^{11} = (1+x)^3(1+x)^8$

General Term of $(1+x)^{11}$

$T_k = {}^{11}C_k \, 1^{11-k} x^k$

$\therefore {}^{11}C_5$ is the coefficient of x^5

$(1+x)^3 = {}^3C_0 + {}^3C_1 x + {}^3C_2 x^2 + {}^3C_3 x^3$

$(1+x)^8 = {}^8C_0 + {}^8C_1 x + {}^8C_2 x^2 + {}^8C_3 x^3 + {}^8C_4 x^4 + {}^8C_5 x^5 + \cdots$

$\therefore$ coefficient of x^5 in $(1+x)^3(1+x)^8$

$= {}^3C_0 \times {}^8C_5 + {}^3C_1 \times {}^8C_4 + {}^3C_2 \times {}^8C_3 + {}^3C_3 \times {}^8C_2$

$= {}^8C_5 + {}^3C_1 \times {}^8C_4 + {}^3C_2 \times {}^8C_3 + {}^8C_2$

Equating coefficients

$\binom{11}{5} = \binom{8}{5} + \binom{3}{1}\binom{8}{4} + \binom{3}{2}\binom{8}{3} + \binom{8}{2}$

1 mark for correct expansion of the LHS and knowing ${}^{11}C_5$ is the coefficient of x^5

2 marks for correct expansion of the RHS and equating coefficients of x^5 correctly

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Examination:..................................

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Question 13

1 mark for writing correct definite integral
1 mark for correctly expanding the integrand
1 mark for correct integration
1 mark for correct answer in simplest exact form

$$\int_{-1}^{2} \pi (2^x - 2)^2 \, dx$$

$$= \pi \int_{-1}^{2} 2^{2x} - 4 \times 2^x + 4 \, dx$$

$$= \pi \left[\frac{1}{2\ln 2} \times 2^{2x} - \frac{4}{\ln 2} \times 2^x + 4x \right]_{-1}^{2}$$

$$= \pi \left[\left(\frac{1}{2\ln 2} \times 2^4 - \frac{4}{\ln 2} \times 2^2 + 8 \right) - \left(\frac{1}{2\ln 2} \times 2^{-2} - \frac{4}{\ln 2} \times 2^{-1} - 4 \right) \right]$$

$$= \pi \left[\frac{1}{2\ln 2} \times 16 - \frac{4}{\ln 2} \times 4 + 8 - \frac{1}{2\ln 2} \times \frac{1}{4} + \frac{4}{\ln 2} \times \frac{1}{2} + 4 \right]$$

$$= \pi \left[\frac{1}{2\ln 2} \left(16 - \frac{1}{4} \right) - \frac{4}{\ln 2} \left(4 - \frac{1}{2} \right) + 12 \right]$$

$$= \pi \left[\frac{63}{8\ln 2} - \frac{14}{\ln 2} + 12 \right]$$

$$= \pi \left[-\frac{49}{8\ln 2} + 12 \right]$$

PE1

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Examination:..................................

Student Number

(b) At time T, $\underset{\sim}{r_A}(T) = r_B(T)$

Hence $VT\cos\alpha = UT\cos\beta$ ①

$-\frac{1}{2}gT^2 + VT\sin\alpha = h - \frac{1}{2}gT^2 + UT\sin\beta$ ②

$\therefore$ ① $\Rightarrow V\cos\alpha = U\cos\beta$ ③

1 mark for $V\cos\alpha = U\cos\beta$

1 mark for $VT\sin\alpha = h + UT\sin\beta$

2 marks for writing correct pair of simultaneous equations to model equal position vectors and solve for T

② $\Rightarrow VT\sin\alpha = h + UT\sin\beta$ ④

③ $\times T\sin\alpha$

$VT\sin\alpha\cos\alpha = UT\cos\beta\sin\alpha$ ⑤

④ $\times \cos\alpha$

$VT\sin\alpha\cos\alpha = h\cos\alpha + UT\sin\beta\cos\alpha$ ⑥

⑥ - ⑤

$h\cos\alpha + UT\sin\beta\cos\alpha - UT\cos\beta\sin\alpha = 0$

$UT(\sin\beta\cos\alpha - \cos\beta\sin\alpha) = -h\cos\alpha$

$UT(\sin\alpha\cos\beta - \sin\beta\cos\alpha) = h\cos\alpha$

$UT(\sin(\alpha-\beta)) = h\cos\alpha$

$\therefore T = \dfrac{h\cos\alpha}{U\sin(\alpha-\beta)}$

(c) (i) LHS $= \tan 3\theta$

$= \tan(2\theta + \theta)$

$= \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$

$= \dfrac{\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right) + \tan\theta}{1 - \tan\theta \times \dfrac{2\tan\theta}{1-\tan^2\theta}}$

$= \dfrac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{1-\tan^2\theta - 2\tan^2\theta}$

$= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$=$ RHS

1 mark for writing $\tan 3\theta$ in terms of $\tan 2\theta$ and $\tan\theta$

1 mark for using appropriate trig identitie to obtain the result

(ii) Let $\tan\theta = x$

$\therefore \dfrac{3x - x^3}{1 - 3x^2} = \sqrt{3}$

$\Rightarrow \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$\tan 3\theta = \sqrt{3}$

$3\theta = \dfrac{\pi}{3}, \dfrac{4\pi}{3}, \dfrac{7\pi}{3}$

$\theta = \dfrac{\pi}{9}, \dfrac{4\pi}{9}, \dfrac{7\pi}{9}$

$\therefore x = \tan\dfrac{\pi}{9}, \tan\dfrac{4\pi}{9}$, and $\tan\dfrac{7\pi}{9}$

1 mark for making substitution for x

1 mark for evaluating $\tan 3\theta = \sqrt{3}$

1 mark for final answers

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(iii) $\frac{3x - x^3}{1 - 3x^2} = \sqrt{3}$

$\sqrt{3} - 3\sqrt{3}x^2 = 3x - x^3$

$x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$

Since $\tan\frac{\pi}{9}$, $\tan\frac{4\pi}{9}$ and $\tan\frac{7\pi}{9}$ are the roots of this equation

$\therefore \tan\frac{\pi}{9} \times \tan\frac{4\pi}{9} \times \tan\frac{7\pi}{9} = -\frac{d}{a} = -\sqrt{3}$

1 mark for obtaining the correct polynomial equation
1 mark for using the relationships between roots and coefficients of cubic equation to deduce the result

Question 14

a) (i) $x = \cos 2\theta$

$dx = -2\sin 2\theta \, d\theta$

$x = 1 \quad \theta \Rightarrow 0$

$x = 0 \quad \theta = \frac{\pi}{4}$

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$$

$$= -\int_{\frac{\pi}{4}}^{0} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \times 2\sin 2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \times 4\sin\theta\cos\theta \, d\theta$$

1 mark for correct substitution

1 mark for using appropriate trig identities to simplify integrand

$$= \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cancel{\cos\theta}} \times 4\sin\theta \cancel{\cos\theta} \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 4\sin^2\theta \, d\theta$$

(ii) $$\int_0^{\frac{\pi}{4}} 4\sin^2\theta \, d\theta$$

1 mark for correct integration

1 mark for correct answer

$$= 2\int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\, d\theta$$

$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - (0)\right]$$

$$= \frac{\pi}{2} - 1$$

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(b) $2x^3 - x^2 + x - 1 = 0$.

Let α, β, γ be the roots

$\alpha + \beta + \gamma = \frac{1}{2}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{1}{2}$

$\alpha\beta\gamma = \frac{1}{2}$

1 mark for correct roots and coefficients relations.

1 mark for correct process to show the answer

(i) $\alpha^2 + \beta^2 + \gamma^2$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (\frac{1}{2})^2 - 2 \times \frac{1}{2}$

$= \frac{1}{4} - 1$

$= -\frac{3}{4}$

ii) sub α, β and γ into $P(x) = 2x^3 - x^2 + x - 1 = 0$.

$2\alpha^3 - \alpha^2 + \alpha - 1 = 0$ ①

$2\beta^3 - \beta^2 + \beta - 1 = 0$ ②

$2\gamma^3 - \gamma^2 + \gamma - 1 = 0$ ③

1 mark for correct working.

1 mark for correct process to show the answer

$2(\alpha^3+\beta^3+\gamma^3) - (\alpha^2+\beta^2+\gamma^2) + (\alpha+\beta+\gamma) - 3 = 0$

$2(\alpha^3+\beta^3+\gamma^3) + \frac{3}{4} + \frac{1}{2} - 3 = 0$.

$2(\alpha^3+\beta^3+\gamma^3) = \frac{7}{4}$

$\alpha^3+\beta^3+\gamma^3 = \frac{7}{8}$

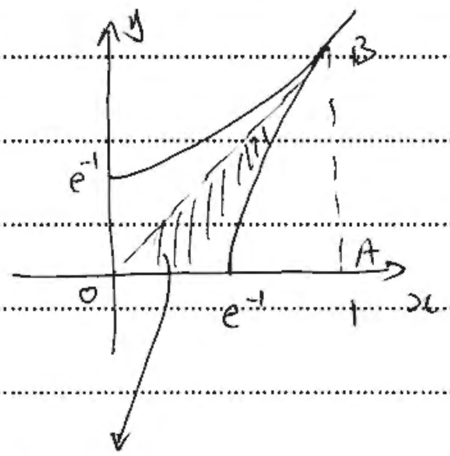
(c) (i) $\frac{d}{dx}(x\ln x)$

$= 1 \times \ln x + x \times \frac{1}{x}$

$= \ln x + 1$

1 mark for using product rule to obtain required result

(ii)



A = Area of the triangle OAB − Area under the curve $y=f(x)$

$= \frac{1}{2} \times 1 \times 1 - \int_{e^{-1}}^{1} (1+\ln x)\,dx$

$= \frac{1}{2} - \left[x\ln x\right]_{e^{-1}}^{1}$

$= \frac{1}{2} - \left[1\ln 1 - e^{-1}\ln e^{-1}\right]$

$= \frac{1}{2} - \left[0 + e^{-1}\right]$

$= \frac{1}{2} - \frac{1}{e}$

1 mark for writing correct definite integral associated with the area

1 mark for correct integration

1 mark for finding the required area in simplest exact form

$\therefore$ Area of total shaded part $= 2 \times \left(\frac{1}{2} - \frac{1}{e}\right)$

$= 1 - \frac{2}{e}$

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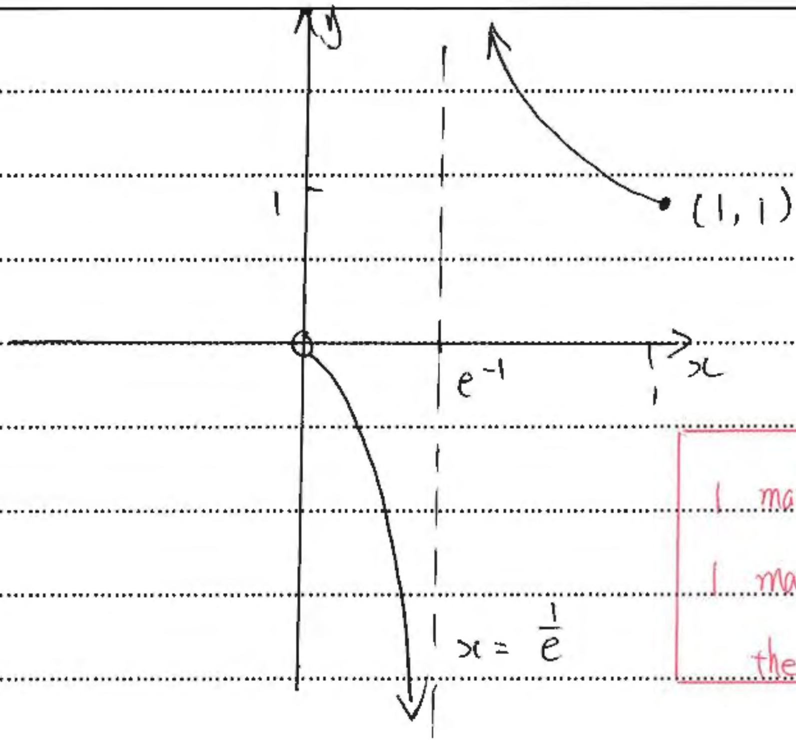
(iii)



1 mark for correct graph.
1 mark for showing All the required detail

(iv) Range is $(-\infty, 0) \cup [1, +\infty)$

1 mark for correctly states range in interval notation

End of Examination.